Zeros of densities and decomposition problem for multidimensional entire characteristic functions of order 2

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Abstract

We consider the entire characteristic functions of order 2 and we prove some decomposition theorems in a multidimensional case. We show that the lack of zeros of the density function is a necessary but not a sufficient (as in the one-dimensional case) condition for a characteristic function to be decomposable. We also find some simple sufficient conditions.

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1 Preliminaries

Entire characteristic functions of order 2 with finite number of zeros were considered by Lukacs. In [1] and [2] he presents theorems related to characteristic functions of the form

$$\varphi(t) = P(t)\exp(A(t)), \qquad t \in \mathbb{R} \tag{1}$$

where P and A are polynomials and A is of order 2. P is a polynomial of an even degree and has a form

$$P(t) = \prod_{j=1}^{d}(1 - \frac{t}{\xi_j})(1 + \frac{t}{\overline{\xi_j}}),$$

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where ξ_j and $\overline{\xi_j}$ are zeros of P. Lukacs has solved decomposition problem for characteristic function of the form (1) where

$$\varphi(t) = P(t)exp\left[-\frac{\sigma^2 t^2}{2}\right], \qquad t \in \mathbb{R}, \tag{2}$$

Then the density function corresponding to the characteristic function (2) has the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} Q(x) \exp\left[-\frac{x^2}{2\sigma^2}\right], \qquad x \in \mathbb{R},$$

where the polynomial Q can be written as

$$Q(x) = \sum_{k=0}^{2n} (-1)^k \lambda_k \sigma^{-k} H_k\left(\frac{x}{\sigma}\right).$$

Here every H_k is the Hermite polynomial of order k and $\lambda_k \in \mathbb{R}$ for $k = 1, 2, ..., 2n$. It is also clear that the polynomial Q must be non-negative for all $x \in \mathbb{R}$. In this paper we will call Q the polynomial *associated* with the characteristic function φ. We have two possibilities for characteristic function of the form (2): either φ is indecomposable or it admits a decomposition

$$\varphi(t) = \varphi_1(t)\varphi_2(t), \qquad x \in \mathbb{R}, \tag{3}$$

where φ_1 and φ_2 are non-trivial characteristic functions. There are again two possibilities in the case where φ of the form (3) is decomposable. From Plucińska (and Lukacs) theorem (see [5]) we have

(a) φ has a normal factor $\varphi_1(t) = \exp\left[-\frac{\sigma_1^2 t^2}{2}\right]$ and then $\varphi_2(t) = P(t)\exp\left[-\frac{\sigma_2^2 t^2}{2}\right]$, where $\sigma_1^2 + \sigma_2^2 = \sigma^2$ or

(b) φ has factors of the form (2) i.e.

$$\varphi_j(t) = P_j(t)\exp\left[-\frac{\sigma_j^2 t^2}{2}\right], \quad j = 1, 2,$$

where $\sigma_1^2 + \sigma_2^2 = \sigma^2$, $P(t) = P_1(t)P_2(t)$, $degP_1 > 0$ and $degP_2 > 0$.

The following theorems can be find in [2].

Theorem 1. *Suppose that the characteristic function φ of the form (2) admits a non-trivial decomposition. Then its associated polynomial Q has no real zeros (see [2] th.7.3.1).*

Theorem 2. *Let φ be the entire characteristic function of the form (2) and suppose that the polynomial associated with φ has no real zeros. Then φ has a normal factor (see (a) above and [2] th.7.3.2).*

From the above theorems we see that the possibility of decomposition of a characteristic function (of the considered type) is equivalent to the statement that the associated polynomial has no real zeros. The aim of this paper is to prove theorems related to the decomposition of multidimensional characteristic functions. We will consider multidimensional polynomial-normal distribution of the form

$$f_{2l}(\mathbf{x}) = \frac{\sqrt{\det \mathbf{A}}}{(2\pi)^{\frac{d}{2}}} p_{2l}(\mathbf{x}) \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{b})^T \mathbf{A}(\mathbf{x}-\mathbf{b})\right), \qquad x \in \mathbb{R}^d,$$

where p_{2l} is non-negative polynomial of order $2l$, $\mathbf{b} \in \mathbb{R}^d$ and $\mathbf{A}$ is a nondegenerate positive $d \times d$ matrix. As we know it from the theory of Fourier transformation (see also Lukacs [2]) the characteristic function of a polynomial-normal distribution is a product of some polynomial and the characteristic function of the normal distribution defined by the same matrix $\mathbf{A}$ and the same vector $\mathbf{b}$. We will prove the following theorem

Theorem 3. *Let the characteristic function φ of d-dimensional polynomial-normal distribution has a non-trivial decomposition $\varphi = \varphi_1\varphi_2$, where φ_1, φ_2 are characteristic functions. Then its associated polynomial Q has no real zeros.*

Proof. Let f_{2l_1} and f_{2l_2} be the densities corresponding to characteristic functions φ_1 and φ_2 respectively.Then by [3], Theorem 2, we have

$$f_{2l_1}(\mathbf{x}) = \frac{\sqrt{\det \mathbf{A}_1}}{(2\pi)^{\frac{d}{2}}} p_{2l_1}(\mathbf{x}) \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{b})^T \mathbf{A}_1(\mathbf{x}-\mathbf{b})\right),$$

$$f_{2l_2}(\mathbf{x}) = \frac{\sqrt{\det \mathbf{A}_2}}{(2\pi)^{\frac{d}{2}}} p_{2l_2}(\mathbf{x}) \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{b}_2)^T \mathbf{A}_2(\mathbf{x}-\mathbf{b}_2)\right), \tag{4}$$

where p_{2l_1}, p_{2l_2} are non-negative polynomials determinated by the zeros of φ_1 and φ_2 respectively, $\mathbf{A_1}$ and $\mathbf{A_2}$ are non-degenerate, positive defined $d \times d$ matrixes and $\mathbf{b_1}, \mathbf{b_2} \in \mathbb{R}^{\mathbf{d}}$. From (4), from the equality $\varphi(\mathbf{t}) = \varphi_1(\mathbf{t})\varphi_2(\mathbf{t})$ and from Borel theorem for Fourier transformation we have

$$f_{2l}(\mathbf{x}) = \int_{\mathbb{R}^d} f_{2l_2}(\mathbf{x}-\mathbf{y}) f_{2l_1}(\mathbf{y}) d\mathbf{y} =$$

$$= \frac{\sqrt{\det \mathbf{A}_1 \det \mathbf{A}_2}}{(2\pi)^d} \int_{\mathbb{R}^d} p_{2l_2}(\mathbf{x}-\mathbf{y}) p_{2l_1}(\mathbf{y}) \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{y}-\mathbf{b}_2)^T \mathbf{A}_2(\mathbf{x}-\mathbf{y}-\mathbf{b}_2)\right) \times$$

$$\times \exp\left(-\frac{1}{2}(\mathbf{y}-\mathbf{b}_1)^T \mathbf{A}_1(\mathbf{y}-\mathbf{b}_1)\right) d\mathbf{y}, \qquad x \in \mathbb{R}^d.$$

(see Maurin [4]).

Let us assume that the polynomial Q takes value zero at a point $\mathbf{x_0} \in \mathbb{R}^d$.

Then $f_{2l}(\mathbf{x}_0) = 0$, so

$$\frac{\sqrt{\det \mathbf{A}_1 \det \mathbf{A}_2}}{(2\pi)^d} \int\limits_{\mathbb{R}^d} p_{2l_2}(\mathbf{x}_0 - \mathbf{y}) p_{2l_1}(\mathbf{y}) \exp\left(-\frac{1}{2}(\mathbf{x}_0 - \mathbf{y} - \mathbf{b}_2)^T \mathbf{A}_2 (\mathbf{x}_0 - \mathbf{y} - \mathbf{b}_2)\right) \times$$

$$\times \exp\left(-\frac{1}{2}(\mathbf{y} - \mathbf{b}_1)^T \mathbf{A}_1 (\mathbf{y} - \mathbf{b}_1)\right) d\mathbf{y} = 0.$$

The above equality holds only in case where the integrand function is equal zero, because it is non-negative and continuous. Then we have

$$\bigwedge_{\mathbf{y} \in \mathbb{R}^d} p_{2l_2}(\mathbf{x}_0 - \mathbf{y}) p_{2l_1}(\mathbf{y}) = 0.$$

So

$$\bigwedge_{\mathbf{y} \in \mathbb{R}^d} p_{2l_2}(\mathbf{x}_0 - \mathbf{y}) = 0 \ \vee p_{2l_1}(\mathbf{y}) = 0. \tag{5}$$

Since the polynomials p_{2l_1} and p_{2l_2} are not equal zero, the sets

$$\left\{\mathbf{y} \in \mathbb{R}^d : p_{2l_2}(\mathbf{x} - \mathbf{y}) = 0\right\} \text{ and } \left\{\mathbf{y} \in \mathbb{R}^d : p_{2l_1}(\mathbf{y}) = 0\right\}$$

are closed and their interiors in $\mathbb{R}^d$ are empty. Then their sum has the empty interior (the trivial case of Baire theorem) and the condition (5) is not fulfilled. Hence one of the polynomials p_{2l_1} or p_{2l_2} is equal zero - contradiction.

Finally, the polynomial Q associated with the given characteristic function has no real zeros. □

Now we present an example which shows that Theorem 2 is not true in multidimensional case.

Example 4. Let (X_1, X_2) be the 2-dimensional random variable with the density

$$f(x_1, x_2) = \frac{1}{6\pi}\left[(x_1 x_2 - 1)^2 + x_2^2\right] \exp\left\{-\frac{1}{2}\left(x_1^2 + x_2^2\right)\right\} =$$

$$= \frac{1}{2\pi} p_4(x_1, x_2) \exp\left\{-\frac{1}{2}\left(x_1^2 + x_2^2\right)\right\}.$$

Then the characteristic function has the form

$$\varphi(t_1, t_2) = \frac{1}{3}\left[t_1^2 t_2^2 + 2t_1 t_2 - 2t_2^2 - t_1^2 + 3\right] \exp\left\{-\frac{1}{2}\left(t_1^2 + t_2^2\right)\right\}. \tag{6}$$

By [3], Theorem 2 we know that if φ has non-trivial decomposition $\varphi = \varphi_1 \varphi_2$ then φ_1 and φ_2 are characteristic functions of polynomial-normal distributions. Let us first prove that in this case it is a product of characteristic function of polynomial-normal distribution and characteristic function of normal distribution. It is a consequence of the fact that the polynomial

$$P(t_1, t_2) = t_1^2 t_2^2 + 2t_1 t_2 - 2t_2^2 - t_1^2 + 3$$

can not be presented as a product of two polynomials of degree 2.

In fact let us write the polynomial P as

$$W(t_1) = t_1^2\left(t_2^2 - 1\right) + 2t_1t_2 - 2t_2^2 + 3,$$

where for fixed t_2 it is a polynomial of degree 2 in t_1. Then

$$\Delta = 4\left(2t_2^4 - 4t_2^2 + 3\right) > 0 \quad \text{for} \quad t_2 \in \mathbb{R}.$$

So the roots have the form

$$t_{1,2} = \frac{-t_2 \pm \sqrt{2t_2^4 - 4t_2^2 + 3}}{t_2^2 - 1}. \tag{7}$$

If the polynomial P is decomposable there are two possibilities

1. $P(t_1,t_2) = Q_1(t_1,t_2)Q_2(t_1,t_2)$, where

$$Q_j(t_1,t_2) = c_jt_1t_2 + d_jt_1 + e_jt_2 + g_j, \quad j = 1,2.$$

The polynomial P is equal zero in (t_1,t_2) iff one of polynomial Q_j (or both of them) is equal zero. Let

$$c_jt_1t_2 + d_jt_1 + e_jt_2 + g_j = 0.$$

Then for a fixed t_2 we have

$$t_1 = -\frac{e_jt_2 + g_j}{c_jt_2 + d_j}. \tag{8}$$

Formula (8) is not of the form (7). Indeed homography function has only one pole, but roots in (7) are functions with two poles $t_2 = \pm 1$.

2. Let now

$$P(t_1,t_2) = Q_1(t_1,t_2)Q_2(t_1,t_2),$$

where

$$Q_1(t_1,t_2) = a_1t_1^2 + c_1t_1t_2 + d_1t_1 + e_1t_2 + g_1$$

and

$$Q_2(t_1,t_2) = b_2t_2^2 + c_2t_1t_2 + d_2t_1 + e_2t_2 + g_2.$$

For fixed t_2 and $Q_1(t_1,t_2) = 0$ we have

$$\Delta = c_1^2t_2^2 + 2c_1d_1t_2 + d_1^2 - 4a_1e_1t_2 - 4a_1g_1,$$

$$t_1 = \frac{-c_1t_2 - d_1 \pm \sqrt{c_1^2t_2^2 + 2c_1d_1t_2 + d_1^2 - 4a_1e_1t_2 - 4a_1g_1}}{2a_1}. \tag{9}$$

When $Q_2(t_1,t_2) = 0$ then

$$t_1 = -\frac{b_2t_2^2 + e_2t_2 + g_2}{c_2t_2 + d_2}. \tag{10}$$

Functions (9) and (10) have only one pole whereas the function (7) has two poles. Then the polynomial P is indecomposable. We can also obtain this result writing P in the form

$$P(t_1,t_2)=\left(a_1t_1^2+b_1t_2^2+c_1t_1t_2+d_1t_1+e_1t_2+g_1\right)\left(a_2t_1^2+b_2t_2^2+c_2t_1t_2+d_2t_1+e_2t_2+g_2\right),$$

and comparing the coefficients in successive powers of variables t_1,t_2.

Hence, if the characteristic function of the form (6) is decomposable, then it is a product of the characteristic function of a polynomial-normal distribution and the characteristic function of a normal distribution. Let

$$\varphi(t_1,t_2)=\frac{1}{3}\left[t_1^2t_2^2+2t_1t_2-2t_2^2-t_1^2+3\right]\exp\left\{-\frac{1}{2}\left(a_{11}t_1^2+2a_{12}t_1t_2+a_{22}t_2^2\right)\right\}\times$$
$$\times\exp\left\{-\frac{1}{2}\left(t_1^2\left(1-a_{11}\right)-2a_{12}t_1t_2+\left(1-a_{22}\right)t_2^2\right)\right\},$$

be such a decomposition. We will show that it is not possible because the function

$$\varphi_1(t_1,t_2)=\frac{1}{3}\left[t_1^2t_2^2+2t_1t_2-2t_2^2-t_1^2+3\right]\exp\left\{-\frac{1}{2}\left(a_{11}t_1^2+2a_{12}t_1t_2+a_{22}t_2^2\right)\right\}, \tag{11}$$

where

$$a_{11},a_{22}>0,\quad a_{11}a_{22}-a_{12}^2>0,\quad 1-a_{11},1-a_{22}>0,\quad (1-a_{11})(1-a_{22})-a_{12}^2>0,$$

could not be a characteristic function of polynomial-normal distribution.

The density function for characteristic function of the form (11) is

$$f_1(x_1,x_2)=\frac{1}{2\pi}\iint\limits_{\mathbb{R}^2}\frac{1}{3}\left[t_1^2t_2^2+2t_1t_2-2t_2^2-t_1^2+3\right]\times$$
$$\times\exp\left\{-\frac{1}{2}\left(a_{11}t_1^2+2a_{12}t_1t_2+a_{22}t_2^2\right)\right\}\exp\left\{-it_1x_1-it_2x_2\right\}dt_2dt_1=$$
$$=\frac{1}{2\pi\sqrt{a_{11}a_{22}-a_{12}^2}}\frac{1}{3}\left[\frac{a_{12}^2}{\left(a_{11}a_{22}-a_{12}^2\right)^2}X^4-\frac{2a_{12}}{\left(a_{11}a_{22}-a_{12}^2\right)^{\frac{3}{2}}}X^3Y+\right.$$
$$+\frac{1}{a_{11}a_{22}-a_{12}^2}X^2Y^2+\left(\frac{6a_{12}}{\left(a_{11}a_{22}-a_{12}^2\right)^{\frac{3}{2}}}-\frac{4a_{12}}{a_{22}\sqrt{a_{11}a_{22}-a_{12}^2}}-\frac{2}{\sqrt{a_{11}a_{22}-a_{12}^2}}\right)XY+$$
$$+\left(a_{22}+2a_{12}+\frac{a_{12}^2}{a_{22}}-1-\frac{6a_{12}^2}{a_{11}a_{22}-a_{12}^2}\right)\frac{X^2}{a_{11}a_{22}-a_{12}^2}+\left(\frac{2}{a_{22}}-\frac{1}{a_{11}a_{22}-a_{12}^2}\right)Y^2+$$
$$\left.+3-\frac{2}{a_{22}}+\frac{3a_{12}^2}{a_{11}a_{22}-a_{12}^2}+\frac{a_{22}}{a_{11}a_{22}-a_{12}^2}\left(-1-\frac{2a_{12}}{a_{22}}-\frac{a_{12}^2}{a_{22}^2}+\frac{1}{a_{22}}\right)\right]\times$$
$$\times\exp\left\{-\frac{1}{2}\left(X^2+Y^2\right)\right\}=\frac{1}{2\pi\sqrt{a_{11}a_{22}-a_{12}^2}}\frac{1}{3}\widetilde{Q}(X,Y)\exp\left\{-\frac{1}{2}\left(X^2+Y^2\right)\right\},$$

where

$$X = \frac{x_1 - \frac{a_{12}}{a_{22}} x_2}{\sqrt{a_{11} - \frac{a_{12}^2}{a_{22}}}}, \qquad Y = \frac{x_2}{\sqrt{a_{22}}}.$$

Let us see that for $a_{12} = 0$ the coefficient of X^4 disappear and the coefficient of X^2 is negative. Then for $Y = 0$ and for X large enough the polynomial $\widetilde{Q}$ has negative values at points $(X, 0)$ – contradiction.

Let us consider the case when $a_{12} \neq 0$. Now we prove that the polynomial in above density function is non-positive. First suppose that $a_{12} > 0$ and substitute

$$T = \sqrt{\frac{a_{12}}{a_{11}a_{22} - a_{12}^2}} X,$$

$a_{12} > 0$. Then

$$\widetilde{Q}(X,Y) = T^4 - \frac{2}{\sqrt{a_{12}}} T^3 Y + \frac{T^2Y^2}{a_{12}} + \left(\frac{6\sqrt{a_{12}}}{a_{11}a_{22} - a_{12}^2} - \frac{4\sqrt{a_{12}}}{a_{22}} - \frac{a_{11}}{a_{22}\sqrt{a_{12}}} \right) TY +$$
$$+ \left(\frac{a_{22}}{a_{12}} + 2 + \frac{a_{12}}{a_{22}} - \frac{1}{a_{12}} - \frac{6a_{12}}{a_{11}a_{22} - a_{12}^2} \right) T^2 + \left(\frac{2}{a_{22}} - \frac{1}{a_{11}a_{22} - a_{12}^2} \right) Y^2 +$$
$$+3 - \frac{2}{a_{22}} + \frac{3a_{12}^2}{a_{11}a_{22} - a_{12}^2} + \frac{a_{22}}{a_{11}a_{22} - a_{12}^2} \left(-1 - \frac{2a_{12}}{a_{22}} - \frac{a_{12}^2}{a_{22}^2} + \frac{1}{a_{22}} \right) =: Q_1(T,Y).$$

Let us denote

$$\widetilde{p}_4(x_1, x_2) = (x_1x_2 - 1)^2 + x_2^2.$$

Thus

$$Q_1(T,Y) = \widetilde{p}_4 \left(T, T - \frac{Y}{\sqrt{a_{12}}} \right) + \left(3 + \frac{a_{22}}{a_{12}} + \frac{a_{12}}{a_{22}} - \frac{1}{a_{12}} - \frac{6a_{12}}{a_{11}a_{22} - a_{12}^2} \right) T^2 +$$
$$+ \left(\frac{2}{a_{22}} - \frac{1}{a_{11}a_{22} - a_{12}^2} - \frac{1}{a_{12}} \right) Y^2 + \left(\frac{6\sqrt{a_{12}}}{a_{11}a_{22} - a_{12}^2} - \frac{2}{\sqrt{a_{12}}} - \frac{4\sqrt{a_{12}}}{a_{22}} \right) TY +$$
$$+2 - \frac{2}{a_{22}} + \frac{3a_{12}^2}{a_{11}a_{22} - a_{12}^2} + \frac{a_{22}}{a_{11}a_{22} - a_{12}^2} \left(-1 - \frac{2a_{12}}{a_{22}} - \frac{a_{12}^2}{a_{22}^2} + \frac{1}{a_{22}} \right).$$

Let us substitute

$$Y_n = \left(n - \frac{1}{n} \right) \sqrt{a_{12}} \quad \text{i} \quad T_n = n.$$

Thus

$$T_n - \frac{Y_n}{\sqrt{a_{12}}} = \frac{1}{n}$$

and

$$\widetilde{p}_n \left(T_n, T_n - \frac{Y_n}{\sqrt{a_{12}}} \right) = \frac{1}{n^2}.$$

Hence

$$Q_1(T_n, Y_n) = \widetilde{p}_4\left(T_n, T_n - \frac{Y_n}{\sqrt{a_{12}}}\right) + \left(\frac{a_{22}}{a_{12}} + 3 + \frac{a_{12}}{a_{22}} - \frac{1}{a_{12}} - \frac{6a_{12}}{a_{11}a_{22} - a_{12}^2}\right) n^2 +$$

$$+ \left(\frac{2}{a_{22}} - \frac{1}{a_{12}} - \frac{1}{a_{11}a_{22} - a_{12}^2}\right) a_{12}\left(n - \frac{1}{n}\right)^2 +$$

$$+ \left(\frac{6\sqrt{a_{12}}}{a_{11}a_{22} - a_{12}^2} - \frac{2}{\sqrt{a_{12}}} - \frac{4\sqrt{a_{12}}}{a_{22}}\right) \sqrt{a_{12}}\left(n^2 - 1\right) +$$

$$+ 2 - \frac{2}{a_{22}} + \frac{1}{a_{11}a_{22} - a_{12}^2}\left(1 - a_{22} - 2a_{12} - \frac{a_{12}^2}{a_{22}}\right) + \frac{3a_{12}^2}{(a_{11}a_{22} - a_{12}^2)^2}.$$

Then the coefficient B_{n^2} of the variable n^2 is equal to

$$B_{n^2} = \frac{a_{22}}{a_{12}} - \frac{a_{12}}{a_{22}} - \frac{1}{a_{12}} - \frac{a_{12}}{a_{11}a_{22} - a_{12}^2}.$$

We will show that B_{n^2} is negative.

Let us assume first that B_{n^2} is not negative. Then

$$\frac{a_{22}}{a_{12}} \geqslant \frac{a_{12}}{a_{22}} + \frac{1}{a_{12}} + \frac{a_{12}}{a_{11}a_{22} - a_{12}^2}.$$

Successive calculation gives

$$a_{22}^2 \geqslant a_{22} + a_{12}^2 + \frac{a_{12}^2 a_{22}}{a_{11}a_{22} - a_{12}^2}$$

(because $a_{12} > 0$),

$$a_{12}^2\left(a_{12}^2 - a_{22}^2\right) \geqslant a_{11}a_{22}\left(a_{22} + a_{12}^2 - a_{22}^2\right) \tag{12}$$

Since $a_{22} + a_{12}^2 - a_{22}^2 = a_{22}(1 - a_{22}) + a_{12}^2 > 0$ and $a_{11}a_{22} > a_{12}^2$ we have

$$a_{11}a_{22}\left(a_{22} + a_{12}^2 - a_{22}^2\right) > a_{12}^2\left(a_{22} + a_{12}^2 - a_{22}^2\right).$$

Then by (12) we obtain

$$a_{12}^2 - a_{22}^2 \geqslant a_{22} + a_{12}^2 - a_{22}^2,$$

and therefore

$$0 \geqslant a_{22}.$$

The last inequality is false, because $a_{22} > 0$. Then $B_{n^2} < 0$ and there exists such $n \in \mathbb{N}$, that $Q_1(T_n, Y_n) < 0$. It means that $f_1(x_{1,n}, x_{2,n}) < 0$ at $(x_{1,n}, x_{2,n})$ corresponding to (T_n, Y_n) – contradiction.

When $a_{12} < 0$ we shall substitute in the above considerations a_{12} by $-a_{12}$. We obtain the same contradiction which means that the function f_1 is not a density function and φ is indecomposable.

Then the decomposition theorem, which is true in one-dimensional case is not true in d-dimensional case, where $d > 1$.

In the next part of this paper we will show that if some condition holds for the associated polynomial of characteristic function φ of d-dimensional polynomial-normal distribution which has no zeros then the characteristic function is decomposable. Let us first prove some auxiliary propositions.

Proposition 5. *Let*

$$Q(\mathbf{x}) = \sum_{|\alpha|\leq 2m} a_\alpha \mathbf{x}^\alpha$$

(where α denotes a multiindex, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$ and $|\alpha| = \alpha_1 + \ldots + \alpha_d$) be the polynomial of d variables, of degree $2m$ with real coefficients, positive on $\mathbb{R}^d$. Let Q satisfies the following condition

$$a := \inf_{\mathbf{x}\in\mathbb{R}^d} \frac{Q\left(x_1, ..., x_d\right)}{1 + \sum\limits_{|\alpha|\leq 2m} |\mathbf{x}^\alpha|} > 0. \tag{13}$$

Then there exists $\varepsilon > 0$ such that, if

$$W(\mathbf{x}) = \sum_{|\alpha|\leq 2m} b_\alpha \mathbf{x}^\alpha, \qquad \mathbf{x} \in \mathbb{R}^d$$

and for every α

$$|a_\alpha - b_\alpha| < \varepsilon,$$

then the polynomial W has only positive values on $\mathbb{R}^d$.

Proof. If $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$, then

$$W(\mathbf{x}) = \sum_{|\alpha|\leq 2m} b_\alpha \mathbf{x}^\alpha = \sum_{|\alpha|\leq 2m} \left(b_\alpha - a_\alpha\right) \mathbf{x}^\alpha + \sum_{|\alpha|\leq 2m} a_\alpha \mathbf{x}^\alpha \geq \tag{14}$$

$$\geq Q(\mathbf{x}) - \sum_{|\alpha|\leq 2m} |b_\alpha - a_\alpha| \, |\mathbf{x}^\alpha| \geq Q(\mathbf{x}) - \left(\sum_{|\alpha|\leq 2m} \varepsilon \, |\mathbf{x}^\alpha| + \varepsilon\right).$$

From (13) we have

$$\frac{Q\left(\mathbf{x}\right)}{1 + \sum\limits_{|\alpha|\leq 2m} |\mathbf{x}^\alpha|} \geq a,$$

so

$$Q\left(\mathbf{x}\right) \geq a \left(1 + \sum_{|\alpha|\leq 2m} |\mathbf{x}^\alpha|\right). \tag{15}$$

From (14) and (15) we have

$$W(\mathbf{x}) \geq a \left(1 + \sum_{|\alpha|\leq 2m} |\mathbf{x}^\alpha|\right) - \varepsilon \left(1 + \sum_{|\alpha|\leq 2m} |\mathbf{x}^\alpha|\right) = (a - \varepsilon) \left(1 + \sum_{|\alpha|\leq 2m} |\mathbf{x}^\alpha|\right).$$

If $\varepsilon < a$, then $W(\mathbf{x}) > 0$, and the theorem is proved.

□

Let us see that for every $\mathbf{x} \in \mathbb{R}^d$ the following equalities and inequalities hold

$$1 + \sum_{j=1}^{d} |x_j|^{2m} \leq 1 + \sum_{|\alpha| \leq 2m} |\mathbf{x}^\alpha| \tag{16}$$

(the left hand side is the addend of the right hand side);

$$1 + \sum_{|\alpha| \leq 2m} |\mathbf{x}^\alpha| = 1 + \sum_{|\alpha| \leq 2m} |x_1|^{\alpha_1} .. |x_d|^{\alpha_d} \leq 1 + \sum_{|\alpha| \leq 2m} \left(1 + \sum_{j=1}^{d} |x_j|\right)^{\alpha_1} ... \left(1 + \sum_{j=1}^{d} |x_j|\right)^{\alpha_d} \leq$$

$$\leq \left(1 + \sum_{j=1}^{d} |x_j|\right)^{2m} + \sum_{|\alpha| \leq 2m} \left(1 + \sum_{j=1}^{d} |x_j|\right)^{2m} = \left(1 + E_{2m}^d\right) \left(1 + \sum_{j=1}^{d} |x_j|\right)^{2m}, \tag{17}$$

where E_{2m}^d is the number of elements of the set $\left\{\alpha \in \mathbb{Z}_+^d : |\alpha| \leq 2m\right\}$;

$$\begin{cases} 1 \leq \sqrt[2m]{1 + \sum\limits_{j=1}^{d} |x_j|^{2m}}, \\ |x_j| \leq \sqrt[2m]{1 + \sum\limits_{k=1}^{d} |x_k|^{2m}} \ \text{ for } \ j = 1, 2, \ldots, d \end{cases}$$

and

$$\left(1 + \sum_{j=1}^{d} |x_j|\right)^{2m} \leq \left(\sqrt[2m]{1 + \sum_{j=1}^{d} |x_j|^{2m}} + \sum_{j=1}^{d} \sqrt[2m]{1 + \sum_{k=1}^{d} |x_k|^{2m}}\right)^{2m} =$$

$$= (1 + d)^{2m} \left(1 + \sum_{j=1}^{d} |x_j|^{2m}\right). \tag{18}$$

From (16), (17) and (18) we have

$$1 + \sum_{j=1}^{d} |x_j|^{2m} \leq 1 + \sum_{|\alpha| \leq 2m} |\mathbf{x}^\alpha| \leq \left(1 + E_{2m}^d\right) \left(1 + \sum_{j=1}^{d} |x_j|\right)^{2m} \leq$$

$$\leq \left(1 + E_{2m}^d\right) (1 + d)^{2m} \left(1 + \sum_{j=1}^{d} |x_j|^{2m}\right).$$

Then we can replace the condition in the above proposition to the one of the following equivalent conditions

$$b := \inf_{\mathbf{x} \in \mathbb{R}^d} \frac{Q\left(x_1, ..., x_d\right)}{1 + \sum\limits_{j=1}^{d} |x_j|^{2m}} > 0, \tag{19}$$

$$c := \inf_{\mathbf{x}\in\mathbb{R}^d} \frac{Q\left(x_1, ..., x_d\right)}{\left(1+\sum\limits_{j=1}^{d}\left|x_j\right|\right)^{2m}} > 0. \tag{20}$$

Proposition 6. *Let* $Q(\mathbf{x}) = \sum\limits_{|\alpha|\leq 2m} a_\alpha \mathbf{x}^\alpha$ *be a polynomial of degree* $2m$ *with real coefficients, positive on* $\mathbb{R}^d$*. Then (13) is equivalent to the following condition*

$$\text{for every } \ 1 \leq j \leq d \qquad \text{we have} \quad \widetilde{a}_{j,2m} > 0, \tag{21}$$

where $\widetilde{a}_{j,2m}$ *is the coefficient of* x_j^{2m} *in* Q*.*

Proof. If e_j is the j-th vector of the standard base in $\mathbb{R}^d$ $\ e_j = \left(\delta_{j1}, \delta_{j2}, ..., \delta_{jd}\right)$, where

$$\delta_{ji} = \begin{cases} 1 & \text{for } \ i = j, \\ 0 & \text{for } \ i \neq j, \end{cases}$$

is the Kronecker's symbol, then for every $r \in \{0, 1, ..., 2m\}$ the sequence $re_j \in \mathbb{Z}_+^d$ is $d-$index of length $|re_j| \leq 2m$. Let us denote

$$\widetilde{a}_{j,r} := a_{re_j}.$$

(13) $\Longrightarrow$ (21) . Let us see that for every $1 \leq j \leq d$ polynomial of one variable

$$Q\left(x_j e_j\right) = \sum_{r=0}^{2m} \widetilde{a}_{j,r} x_j^r, \quad x_j \in \mathbb{R},$$

has positive values on $\mathbb{R}$. Then $a_{j,0} = a_0 = Q(0) > 0$ and $\widetilde{a}_{j,2m} \geq 0$.

Let $\widetilde{a}_{j,2m} = 0$. Then

$$\lim_{x_j\to+\infty} \frac{Q\left(x_j e_j\right)}{1+\left|x_j\right|^{2m}} = \lim_{x_j\to+\infty} \sum_{r=0}^{2m-1} \frac{\widetilde{a}_{j,r} x_j^r}{1+\left|x_j\right|^{2m}} = 0$$

and

$$\inf_{\mathbf{x}\in\mathbb{R}^d} \frac{Q\left(\mathbf{x}\right)}{1+\sum\limits_{j=1}^{d}\left|x_j\right|^{2m}} = 0.$$

The last equality contradicts the condition (19), so it contradicts (13). Then we must have $\widetilde{a}_{j,2m} > 0$.

(13) $\Longleftarrow$ (21) . Let us write the polynomial Q in the form

$$Q(\mathbf{x}) = \sum_{j=1}^{d} \widetilde{a}_{j,2m} x_j^{2m} + \sum_{|\alpha|<2m} a_\alpha \mathbf{x}^\alpha, \qquad \mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d,$$

and let us denote $\ ||\mathbf{x}||_1 = \sum\limits_{j=1}^{d} |x_j|$ (l^1-norm in $\mathbb{R}^d$). From consideration concerning inequality (17) follows that for every $\alpha \in \mathbb{Z}_+^d$ such that $|\alpha| = r$ the

following equality holds

$$\frac{|\mathbf{x}^\alpha|}{\left(1+\sum\limits_{j=1}^{d}|x_j|\right)^{2m}} \leq \frac{\left(1+\sum\limits_{j=1}^{d}|x_j|\right)^{r}}{\left(1+\sum\limits_{j=1}^{d}|x_j|\right)^{2m}} = \frac{1}{\left(1+||\mathbf{x}||_1\right)^{2m-r}}.$$

Then taking (18) we have

$$\frac{\left|\sum\limits_{|\alpha|<2m} a_\alpha \mathbf{x}^\alpha\right|}{1+\sum\limits_{j=1}^{d}|x_j|^{2m}} \leq \frac{(1+d)^{2m}\left|\sum\limits_{|\alpha|<2m} a_\alpha \mathbf{x}^\alpha\right|}{\left(1+||\mathbf{x}||_1\right)^{2m}} \leq \sum_{|\alpha|<2m} \frac{|a_\alpha|}{\left(1+||\mathbf{x}||_1\right)^{2m-|\alpha|}}(1+d)^{2m}$$

and consistently

$$\lim_{||\mathbf{x}||_1\to+\infty} \frac{\sum\limits_{|\alpha|<2m} a_\alpha \mathbf{x}^\alpha}{1+\sum\limits_{j=1}^{d}|x_j|^{2m}} = \lim_{||\mathbf{x}||_1\to+\infty} \frac{\sum\limits_{|\alpha|<2m} a_\alpha \mathbf{x}^\alpha}{\left(1+||\mathbf{x}||_1\right)^{2m}} = 0.$$

Let us denote

$$b_0 := \min_{1\leq j\leq d} \widetilde{a}_{j,2m}.$$

Then

$$\frac{\sum\limits_{j=1}^{d}\widetilde{a}_{j,2m}x_j^{2m}}{1+\sum\limits_{j=1}^{d}x_j^{2m}} \geq \frac{b_0\sum\limits_{j=1}^{d}x_j^{2m}}{1+\sum\limits_{j=1}^{d}x_j^{2m}} \xrightarrow[||\mathbf{x}||_1\to+\infty]{} b_0$$

(Because in $\mathbb{R}^d$ l^1-norm $||\cdot||_1$ is equivalent $l^{2m}-$norm $||\cdot||_{2m}$, so $\sum\limits_{j=1}^{d}x_j^{2m} = \left(||\mathbf{x}||_{2m}\right)^{2m} \to \infty$, when $||\mathbf{x}||_1 \to \infty$).

Let us take $R>0$ such that for $||\mathbf{x}||_1 > R$ the following inequalities hold

$$\left|\frac{\sum\limits_{|\alpha|<2m} a_\alpha \mathbf{x}^\alpha}{1+\sum\limits_{j=1}^{d}x_j^{2m}}\right| < \frac{b_0}{3},$$

$$\frac{\sum\limits_{j=1}^{d}\widetilde{a}_{j,2m}x_j^{2m}}{1+\sum\limits_{j=1}^{d}x_j^{2m}} \geq \frac{2}{3}b_0.$$

Let us denote by $\overline{K_1}(0,R)$ the closed ball in $\mathbb{R}^d$ with center at $\mathbf{x}=\mathbf{0}$ and radius R with respect to the metric given by the norm $||\cdot||_1$. Then for $\mathbf{x}\in\mathbb{R}^d\backslash\overline{K_1}(0,R)$

$$\frac{Q(\mathbf{x})}{1+\sum\limits_{j=1}^{d}x_j^{2m}}=\frac{\sum\limits_{j=1}^{d}\widetilde{a}_{j,2m}x_j^{2m}}{1+\sum\limits_{j=1}^{d}x_j^{2m}}+\frac{\sum\limits_{|\alpha|<2m}a_\alpha\mathbf{x}^\alpha}{1+\sum\limits_{j=1}^{d}x_j^{2m}}\geq\frac{\sum\limits_{j=1}^{d}\widetilde{a}_{j,2m}x_j^{2m}}{1+\sum\limits_{j=1}^{d}x_j^{2m}}-\left|\frac{\sum\limits_{|\alpha|<2m}a_\alpha\mathbf{x}^\alpha}{1+\sum\limits_{j=1}^{d}x_j^{2m}}\right|>\frac{b_0}{3}.$$

We have

$$\inf_{\mathbf{x}\in\mathbb{R}^d}\frac{Q(\mathbf{x})}{1+\sum\limits_{j=1}^{d}x_j^{2m}}\geq\min\left\{\frac{b_0}{3},\inf_{\mathbf{x}\in\overline{K_1}(0,R)}\frac{Q(\mathbf{x})}{1+\sum\limits_{j=1}^{d}x_j^{2m}}\right\}>0,$$

because the greatest lower bound of the continuous function

$$h(\mathbf{x})=\frac{Q(\mathbf{x})}{1+\sum\limits_{j=1}^{d}x_j^{2m}},\quad\mathbf{x}\in\mathbb{R}^d,$$

on the compact set $\overline{K_1}(0,R)$ is the value of the function at some point $\overline{\mathbf{x}}\in\overline{K_1}(0,R)$, so it is a positive number. It means that the condition (19) holds, so (21) is fulfilled. □

Finally from Proposition 5 and Proposition 6 we have

Theorem 7. *Let*

$$Q(\mathbf{x})=\sum_{|\alpha|\leq 2m}a_\alpha\mathbf{x}^\alpha$$

be the polynomial with real coefficients of degree $2m$*, positive on* $\mathbb{R}^d$ *and such that the condition (21) is satisfied. Then there exists* $\varepsilon>0$ *such that if*

$$W(\mathbf{x})=\sum_{|\alpha|\leq 2m}b_\alpha\mathbf{x}^\alpha,\quad\mathbf{x}\in\mathbb{R}^d,$$

is the polynomial with real coefficients fulfilling inequalities

$$|a_\alpha-b_\alpha|<\varepsilon,$$

for every α*,* $|\alpha|<2m$*, then the polynomial* W *takes only positive values on* $\mathbb{R}^d$*.*

We show that property (21) is invariant with respect to superposition of the polynomial Q with nonsingular linear map of $\mathbb{R}^d$.

Proposition 8. *If a positive polynomial* $Q\in\mathbb{R}[x_1,...,x_d]$ *of degree* $2m$ *satisfies condition (21) and* $\mathcal{A}$ *is an affine isomorphism of* $\mathbb{R}^d$*, then the polynomial* $Q_{\mathcal{A}}:=Q\circ\mathcal{A}\in\mathbb{R}[x_1,...,x_d]$ *is positive and satisfies condition (21).*

Proof. Let us remind that every affine isomorphism in $\mathbb{R}^d$ has the form

$$\mathcal{A}(\mathbf{x}) = F(\mathbf{x}) + \mathbf{b}, \ \ \mathbf{x} \in \mathbb{R}^d,$$

where F is a linear isomorphism and $\mathbf{b} \in \mathbb{R}^d$.

Positivity of the polynomial $Q_{\mathcal{A}}$ is clear. Let us assume that $\mathbf{b} = \mathbf{0}$ and denote by $\left|\left|F^{-1}\right|\right|_1$ the operator norm of the map F^{-1} with respect to l^1-norm $||\cdot||_1$ in $\mathbb{R}^d$. Then $\left|\left|F^{-1}\right|\right|_1 > 0$. Since Q satisfies (21) it satisfies (20), too. We will show that $Q_{\mathcal{A}}$ satisfies (20). Let us consider $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ such that $\mathbf{x} = F(\mathbf{y})$. We have $||\mathbf{y}||_1 = \left|\left|F^{-1}(\mathbf{x})\right|\right|_1 \leq \left|\left|F^{-1}\right|\right|_1 ||\mathbf{x}||_1$ and then

$$\frac{Q_{\mathcal{A}}(\mathbf{y})}{\left(1 + ||\mathbf{y}||_1\right)^{2m}} = \frac{Q(\mathbf{x})}{\left(1 + ||F^{-1}(\mathbf{x})||_1\right)^{2m}} \geq \frac{Q(\mathbf{x})}{\left(1 + ||F^{-1}||_1 \, ||\mathbf{x}||_1\right)^{2m}}.$$

There are two possibilities.

Case 1: $\left|\left|F^{-1}\right|\right|_1 > 1$. Then

$$1 + \left|\left|F^{-1}\right|\right|_1 ||\mathbf{x}||_1 < \left|\left|F^{-1}\right|\right|_1 + \left|\left|F^{-1}\right|\right|_1 ||\mathbf{x}||_1 = \left|\left|F^{-1}\right|\right|_1 \left(1 + ||\mathbf{x}||_1\right)$$

and

$$\frac{Q_{\mathcal{A}}(\mathbf{y})}{\left(1 + ||\mathbf{y}||_1\right)^{2m}} > \frac{Q(\mathbf{x})}{||F^{-1}||_1^{2m} \left(1 + ||\mathbf{x}||_1\right)^{2m}} \geq \frac{c}{||F^{-1}||_1^{2m}} > 0,$$

where c is the least lower bound from condition (20) for Q. Hence

$$\inf_{\mathbf{y} \in \mathbb{R}^d} \frac{Q_{\mathcal{A}}(\mathbf{y})}{\left(1 + ||\mathbf{y}||_1\right)^{2m}} \geq \frac{c}{||F^{-1}||_1^{2m}} > 0.$$

Case 2: $\left|\left|F^{-1}\right|\right|_1 \leq 1$. Then

$$1 + \left|\left|F^{-1}\right|\right|_1 ||\mathbf{x}||_1 \leq 1 + ||\mathbf{x}||_1$$

and

$$\frac{Q_{\mathcal{A}}(\mathbf{y})}{\left(1 + ||\mathbf{y}||_1\right)^{2m}} \geq \frac{Q(\mathbf{x})}{\left(1 + ||\mathbf{x}||_1\right)^{2m}} \geq c > 0.$$

This means that

$$\inf_{\mathbf{y} \in \mathbb{R}^d} \frac{Q_{\mathcal{A}}(\mathbf{y})}{\left(1 + ||\mathbf{y}||_1\right)^{2m}} \geq c > 0.$$

In both cases the polynomial $Q_{\mathcal{A}}$ satisfies condition (20), so it satisfies (21), too.

Let us assume now that $F = id_{\mathbb{R}^d}$ which means that $\mathcal{A}$ is the translation by vector $\mathbf{b} = (b_1, ..., b_d)$. Then from Newton's formula for $(y_i + b_i)^{2m}$ we have that for every $1 \leq j \leq d$ the coefficient of y_i^{2m} in the polynomial $Q_{\mathcal{A}}$ is the same as the coefficient of x_i^{2m} in the polynomial Q. It means that $Q_{\mathcal{A}}$ satisfies condition (21) iff Q satisfies this condition. Thesis of the proposition follows from the fact that every affine isomorphism is a composition of a linear isomorphism and some translation. $\square$

Conclusion 9. *In the above proposition we can substitute implication by equivalence: the polynomial Q satisfies condition (21) iff $Q_{\mathcal{A}}$ satisfies (21).*

Proof. If $\mathcal{A}$ is an isomorphism then $\mathcal{A}^{-1}$ is an isomorphism, too and $Q = (Q_{\mathcal{A}})_{\mathcal{A}^{-1}}$. $\square$

The following theorem is the main result of this paper.

Theorem 10. *Let the function f be the density on $\mathbb{R}^d$ of the form*

$$f(\mathbf{x}) = \frac{\sqrt{\det \mathbf{A}}}{(2\pi)^{\frac{d}{2}}} p_{2l}(\mathbf{x}) \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{b})^T \mathbf{A}(\mathbf{x}-\mathbf{b})\right),$$

where p_{2l} is a positive polynomial of degree $2l$ satisfying (21), and A is a symmetric and positive matrix of dimension $d \times d$. Let φ be the characteristic function of this distribution. Then there are d-dimensional random variables Y and Z – the first with polynomial-normal distribution and the second with the normal distribution such that

$$\varphi = \varphi_Y \varphi_Z,$$

where φ_Y and φ_Z are characteristic functions of Y and Z respectively.

Proof. Let $\widetilde{X}$ be a d-dimensional random variable with density f. Then the random variable $X := \widetilde{X} - \mathbf{b}$ has the density of the form

$$f_X(\mathbf{x}) = \frac{\sqrt{\det \mathbf{A}}}{(2\pi)^{\frac{d}{2}}} p_{2l}(\mathbf{x}+\mathbf{b}) \exp\left(-\frac{1}{2}\mathbf{x}^T \mathbf{A}\mathbf{x}\right).$$

The polynomial p_{2l} is positive (from Theorem 8) and satisfies condition (21). We have

$$\varphi(\mathbf{t}) = \varphi_X(\mathbf{t}) \exp(i\mathbf{b}\cdot\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^d.$$

If $\varphi_X = \varphi_Y \varphi_Z$, where Y and Z are d-dimensional random variables which satisfies condition of the thesis. Then for every $\mathbf{t} \in \mathbb{R}^d$

$$\varphi(\mathbf{t}) = \exp(i\mathbf{b}\cdot\mathbf{t})\,\varphi_Y(\mathbf{t})\varphi_Z(\mathbf{t}) = \varphi_Y(\mathbf{t})\left[\exp(i\mathbf{b}\cdot\mathbf{t})\,\varphi_Z(\mathbf{t})\right] = \varphi_Y(\mathbf{t})\varphi_{\widetilde{Z}}(\mathbf{t}),$$

where $\widetilde{Z} = Z + \mathbf{b}$ is d-dimensional random variable of normal distribution. Hence we may assume that the density f from the thesis of our theorem has the form

$$f(\mathbf{x}) = \frac{\sqrt{\det \mathbf{A}}}{(2\pi)^{\frac{d}{2}}} p_{2l}(\mathbf{x}) \exp\left(-\frac{1}{2}\mathbf{x}^T \mathbf{A}\mathbf{x}\right).$$

Let L be the matrix which reduce $\mathbf{A}$ to a normalized diagonal matrix (to the identity matrix).

Let $\mathbf{X} = L\mathbf{U}$. Then $\mathbf{U} = L^{-1}\mathbf{X}$ and the density function of $\mathbf{U}$ is given by

$$
\begin{aligned}
f_{\mathbf{U}}(\mathbf{u}) &= f_{L^{-1}\mathbf{X}}(\mathbf{u}) = f_{\mathbf{X}}(L\mathbf{u})\,|\det L| = \\
&= \frac{\sqrt{\det \mathbf{A}\,(\det L)^2}}{(2\pi)^{\frac{d}{2}}} p_{2l}(L\mathbf{u}) \exp\left(-\frac{1}{2}(L\mathbf{u})^T \mathbf{A}\,(L\mathbf{u})\right) = \\
&= \frac{\sqrt{\det(L^T\mathbf{A}L)}}{(2\pi)^{\frac{d}{2}}} \widetilde{p_{2l}}(\mathbf{u}) \exp\left(-\frac{1}{2}\mathbf{u}^T(L^T\mathbf{A}L)\mathbf{u}\right) = \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \widetilde{p_{2l}}(\mathbf{u}) \exp\left(-\frac{1}{2}\mathbf{u}^T\mathbf{u}\right).
\end{aligned}
$$

From Proposition 8 the polynomial $\widetilde{p_{2l}}(\mathbf{u}) = p_{2l}(L\mathbf{u})$ is positive and satisfies condition (21). If $\varphi_U = \varphi_{\widetilde{\mathbf{Y}}}\varphi_{\widetilde{\mathbf{Z}}}$, where $\widetilde{\mathbf{Y}}$ has a polynomial-normal distribution and $\widetilde{\mathbf{Z}}$ has a normal distribution then

$$
\varphi_{\mathbf{X}}(\mathbf{t}) = \varphi_{L\mathbf{U}}(\mathbf{t}) = \varphi_{\mathbf{U}}(L^*\mathbf{t}) = \varphi_{\widetilde{\mathbf{Y}}}(L^*\mathbf{t})\,\varphi_{\widetilde{\mathbf{Z}}}(L^*\mathbf{t}) = \varphi_{L\widetilde{\mathbf{Y}}}(\mathbf{t})\,\varphi_{L\widetilde{\mathbf{Z}}}(\mathbf{t}), \qquad \mathbf{t} \in \mathbb{R}^d.
$$

We know that $\mathbf{Y} := L\widetilde{\mathbf{Y}}$ has a polynomial-normal distribution and $\mathbf{Z} := L\widetilde{\mathbf{Z}}$ has a normal distribution. Hence, if the thesis of the theorem holds for the identity matrix I then it holds for any symmetric and positive matrix $\mathbf{A}$.

We must only consider the case when the density from the thesis is given by

$$
f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}} p_{2l}(\mathbf{x}) \exp\left(-\frac{1}{2}\sum_{i=1}^{d} x_i^2\right).
$$

Let φ be the characteristic function of f. Then it can be written in the form

$$
\varphi(\mathbf{t}) = \sum_{|\alpha|\le 2l} \beta_\alpha (i\mathbf{t})^\alpha \exp\left(-\frac{1}{2}\sum_{i=1}^{d} t_i^2\right), \qquad \mathbf{t} \in \mathbb{R}^d,
$$

where $\beta_\alpha \in \mathbb{R}$ (see [2], §7.3). Let

$$
\varphi_\theta(\mathbf{t}) = \sum_{|\alpha|\le 2l} \beta_\alpha (i\mathbf{t})^\alpha \exp\left(-\frac{1}{2}\sum_{i=1}^{d} \theta^2 t_i^2\right), \qquad \mathbf{t} \in \mathbb{R}^d,
$$

where $\theta \in (0,1)$. Then the inverse Fourier transform of φ_θ is equal to

$$
f_\theta(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left[\sum_{|\alpha|\le 2l} \beta_\alpha \frac{1}{\theta^{|\alpha|}} \prod_{j=1}^{d} H_{\alpha_j}\left(\frac{x_j}{\theta}\right)\right] \exp\left(-\frac{1}{2\theta^2}\sum_{j=1}^{d} x_j^2\right),
$$

where α_j denotes the Hermite polynomial of order α_j (see [2], §7.3). It is clear that

$$
\lim_{\theta\to 1^-} f_\theta(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left[\sum_{|\alpha|\le 2l} \beta_\alpha \prod_{j=1}^{d} H_{\alpha_j}(x_j)\right] \exp\left(-\frac{1}{2}\sum_{j=1}^{d} x_j^2\right) = f(\mathbf{x}).
$$

Moreover the coefficients of the polynomial

$$p_{2l,\theta}(\mathbf{x}) = \sum_{|\alpha|\leq 2l} \beta_\alpha \frac{1}{\theta^{|\alpha|}} \prod_{j=1}^{d} H_{\alpha_j}\left(\frac{x_j}{\theta}\right), \qquad \mathbf{x} \in \mathbb{R}^d,$$

converge to the coefficients of the polynomial

$$\sum_{|\alpha|\leq 2l} \beta_\alpha \prod_{j=1}^{d} H_{\alpha_j}(x_j) = p_{2l}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d.$$

Thus, if we write p_{2l} in the form

$$p_{2l}(\mathbf{x}) = \sum_{|\alpha|\leq 2l} a_\alpha \mathbf{x}^\alpha$$

and

$$p_{2l,\theta}(\mathbf{x}) = \sum_{|\alpha|\leq 2l} b_\alpha(\theta)\mathbf{x}^\alpha,$$

then for every $\alpha \in \mathbb{Z}_+^d$, $|\alpha| \leq 2l$ we have

$$\lim_{\theta\to 1^-} b_\alpha(\theta) = a_\alpha.$$

Let $\varepsilon > 0$ be a number from Theorem 7 chosen for the polynomial $Q = p_{2l}$. Then there exists $\delta > 0$ such that if $1-\theta < \delta$, then $|b_\alpha(\theta) - a_\alpha| < \varepsilon$ for $|\alpha| \leq 2l$. If we take $\theta \in (1-\delta, 1)$ then we get nonnegative polynomial $p_{2l,\theta}$ on $\mathbb{R}^d$. Hence f_θ is a density function on $\mathbb{R}^d$ and the random variable $\mathbf{Y}$ has PND_d distribution.

Let

$$f_{\mathbf{Z}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}\left(1-\theta^2\right)^{\frac{d}{2}}} \exp\left(-\frac{1}{2\left(1-\theta^2\right)}\sum_{i=1}^{d} x_i^2\right).$$

Then

$$\varphi_{\mathbf{Z}}(\mathbf{t}) = \exp\left(-\frac{1}{2}\left(1-\theta^2\right)\sum_{i=1}^{d} t_i^2\right).$$

and

$$\varphi_{\mathbf{Y}}(\mathbf{t})\,\varphi_{\mathbf{Z}}(\mathbf{t}) = \varphi_\theta(\mathbf{t})\,\varphi_{\mathbf{Z}}(\mathbf{t}) = \varphi(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^d.$$

This ends the proof of our theorem. □

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